Electron-hole duality and vortex rings in quantum dots

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Abstract

In a quantum-mechanical system, particle-hole duality implies that instead of studying *particles*, we can get equivalent information by studying the missing particles, the so-called *holes*. Using this duality picture for rotating fermion condensates the vortices appear as holes in the Fermi see. Here we predict that the formation of vortices in quantum dots at high magnetic fields causes oscillations in the energy spectrum which can be experimentally observed using accurate tunnelling spectroscopy. We use the duality picture to show that these oscillations are caused by the localisation of vortices in rings.

Vortex formation in quantum liquids has been much discussed in connection with finite rotating condensates of bosonic atoms [1, 2, 3, 4, 5, 6]. More recently, even fermionic atom gases [7, 8, 9] entered the stage. Quite generally, and independent of the underlying statistics, it appears that vortex formation is a universal property of rotating quantal systems with repulsive interaction among the particles [10]. Analogously, theoretical studies have demonstrated that vortices are formed in semiconductor quantum dots at strong external magnetic fields [11], or equivalently at high angular momenta.

Particle-hole duality is frequently used for understanding quantum mechanical many-particle systems: Sometimes the study of the missing particles, i.e. holes, gives a much clearer view of the observed phenomena. For example, when describing the conductance of a semiconductor the use of holes greatly simplifies a complicated physical picture [12]. Related concepts of duality have been used in nuclear physics, where states within some given angular momentum shell can be described either by particles or, equivalently, holes in the same shell [13]. A similar description was used already in the 1930's for the atomic shell model [14]. Recently, particle-hole duality has also been studied in connection with quantum Hall liquids [15, 16, 17].

In this letter we show, starting from exact many-particle theory, and using general arguments that (i) vortices in quantum dots are holes in the polarised Fermi see of electrons, (ii) vortices form a fractional quantum Hall system, (iii) vortices localise in a ring, and (iv) the energy spectrum as a function of magnetic field shows oscillations which can be measured and reveal the number of vortices in the dot.

We first note that in a fairly strong magnetic field the electrons confined by the circular quantum dot form a polarised electron gas. The role of the magnetic field is only to set the electron cloud to rotation [18, 19]. For the present purpose it is thus sufficient to study the development of the many-body state of the electrons with increasing total angular momentum L.

Figure 1 shows the energy spectrum for 20 electrons in a circular two-dimensional quantum dot as a function of L. The spectrum is the result of exact diagonalisation of the Hamiltonian describing the system of interacting electrons [20]. We observe that the lowest-energy state as a function of angular momentum shows periodic oscillations with *successive periods of 2, 3, and 4.* As we will explain below, these oscillations are *signatures of 2, 3 and 4 vortices being localised in a ring.*

Unfortunately, the accuracy of the resonant tunneling measurements available today [21, 22] is not sufficient to uncover these oscillations of the grund state energies with increasing angular momentum and magnetic field. However, the very recent developments in sample preparation and accuracy of tunneling spectroscopy [23] in fact should make it possible to observe these salient features.

To our surprise, we found that the reason behind the periodic oscillations in the energy spectrum is deeply connected with the above mentioned particle-hole duality.

Let us consider a two-dimensional harmonic confinement with N interacting electrons. In a highly rotating state, for which the orbital angular momentum reaches $L = N(N-1)/2$, the so-called maximum density droplet [24] is formed. This state is the finite-size manifestation of the integer quantum Hall liquid. When it occurs, the electron system is fully polarised [25]. Consequently, we can restrict our study to the many-particle physics of *spinless* electrons in high rotational states. We use the configuration interaction (CI) technique to solve the many-particle Schrödinger equation. The many-particle configuration is a Slater determinant of single particle states of the two-dimensional harmonic oscillator. Beyond the maximum density droplet the configuration space can be restricted to the lowest Landau band [19]. Any configuration can then be described with a vector $|n_0, n_1, \cdots, n_M\rangle$, where n_m is the occupation number (0 or 1) for the single particle state with angular momentum m, and M is the the highest single particle angular momentum included in the basis. A state with N electrons will have N *one*'s and $(M-N)$ *zero*'s in the configuration vector, i.e., it will for example be of the form $|\,111110001111\cdots\rangle$. We call the *one*'s particles and the *zero*'s holes. The key issue to be noted here is that *the many-particle problem can be solved by diagonalizing the Hamiltonian matrix either for the particle states or for the hole states*, i.e., replacing each n_m with $1 - n_m$ (the above Fock state will then be $|\,000001110000\cdots\rangle$ for holes)[26]. This duality of the rotational states has two advantages: (i) by *interpreting the holes as vortices*, it gives a natural way to study the vortex-vortex correlations, i.e. the internal structure of the vortex lattice, and (ii) it shows that for small number of vortices, the vortex state can be approximated with the Laughlin wave function for the fractional quantum Hall state, while the electron state is close to the integer quantum Hall state. Although the electrons remain delocalized, the holes will localise in rings resulting in characteristic oscillations of the energy spectra as a function of angular momentum or, equivalently, magnetic field.

Figure 2 shows the vortex-vortex pair correlation for 20 electrons with angular momentum 246. In Fig. 2a the actual calculation for *20 electrons* was performed using $M = 30$ single particle states. *After* diagonalisation of the Hamiltonian matrix, the *many-particle* state was converted to a *few-hole* state, simply by converting each n_m to $1 - n_m$. The vortex-vortex pair correlation clearly shows four localised vortices in the quantum dot, with three maxima each corresponding to a single vortex, and one "missing" vortex due to the reference point (which resembles the "exchange hole" in the particle-particle correlation function). The ring surrounding the localised vortices is due to the six holes at high single particle angular momentum states (for $M = 30$ and $N = 20$ the total number of holes is $M - N = 10$). Figure 2b shows the result of the analogous full many-body calculation for the corresponding situation with *four holes*. Clearly, the vortex structure is identical to that of Fig. 2a, except that now trivially, the hole density outside the electron cloud has vanished. Figure 2 demonstrates clearly that the result of the many-particle state is similar whether it is done with electrons or with holes. A direct comparison of the configuration amplitudes of many-particle states of these two calculations indeed show a very good agreement [26]. We emphasise that when diagonalizing the many-body Hamiltonian in the restricted Hilbert space of the lowest Landau level, the computational work is not easier whether using holes or electrons. However, the duality description makes it possible to determine the vortex texture via the vortex-vortex correlation.

Let us now view the many-particle state for holes in terms of a fractional quantum Hall state. As an example, we use the same state for which we showed the correlation functions in Fig 2b. For four holes, the angular momentum of this state now equals 30. This appears as a small number compared to the angular momentum of the conjugate state for 20 electrons ($L = 246$). In effect, however, the angular momentum of hole states relative to the compact maximum density droplet is larger than that of the electron states due to the much smaller number of "particles". In fact, the hole state can be well approximated by the Laughlin wave function $\Psi = \prod_{i,j}(z_i - z_j)^q \exp(-\sum_k |z_k|^2)$ (where the electron coordinates are described by complex numbers z_i) [27]. Since the angular momentum of such a state is $qN(N - 1)/2$, our four-vortex state with angular momentum 30 corresponds to $q = 5$. More generally, we can argue that a small number of holes (vortices) in a rotating system of a large number of electrons naturally leads to a Laughlin state with large q.

The Laughlin state localises the particles in a lattice. This localisation is the better the

larger the exponent q (see Refs. [19, 27]). For less than six vortices in a quantum dot, these vortices are found to localise in a ring, similarly to the well-known localisation of *particles* in a so-called Wigner molecule [22]. *It is this localisation of vortices that results in the characteristic features of the rotational spectrum.* Rigid rotation of the ring-like 'molecule' formed by the localised vortices becomes possible. For polarised fermions, as in this case, the rigid rotation of the ring of n vortices is allowed only at every nth angular momentum [19, 28, 29]. These angular momenta represent the minima in the energy spectrum, seen in Fig. 1. At intermediate angular momenta, the rigid rotation has to be accompanied with other excitations, like vibrational modes, resulting in higher energies. Consequently, the period of the successive minima gives directly the number of the vortices localised in a ring. The single-vortex state moves towards the center with increasing angular momentum, but does not reach the origin before the two-vortex state becomes the ground state.

When the vortex number increases to six, the localised vortices can have two nearly degenerate textures: A ring of six vortices or a ring of five vortices with one vortex at the centre. As an example we study six vortices in a 30 electron quantum dot. Figure 3 shows the vortex-vortex correlation functions for the two lowest energy states at angular momentum 555, which for the conjugate state for holes corresponds to the Laughlin state with $q = 5$. The lowest energy state shows a ring of six vortices while the nearly degenerate first excited state shows a vortex structure with one vortex at the centre. Classically, particles interacting with repulsive long-range interaction (Coulombic or logarithmic) will have these two stable configurations [22]. Quantum mechanical calculations for spinless fermions also find these internal symmetries, as shown in Fig. 3. As a consequence, the energy spectrum as a function of the angular momentum becomes more complicated when the number of vortices reaches six, due to the two possible periods of rigid rotation [19].

In conclusion, we have shown that the vortex formation in quantum dots most easily can be understood in terms of electron-hole duality. The vortex texture is a natural consequence of the Laughlin-type many-particle wave function for vortices. The localisation of vortices in a ring appears in the rotational spectrum as periodic oscillations of the ground state energy as a function of external magnetic field. This should also be observable experimentally. Finally, we point out that a related duality holds also for trapped bosons and explains the vortex lattices in rotating bose condensates.

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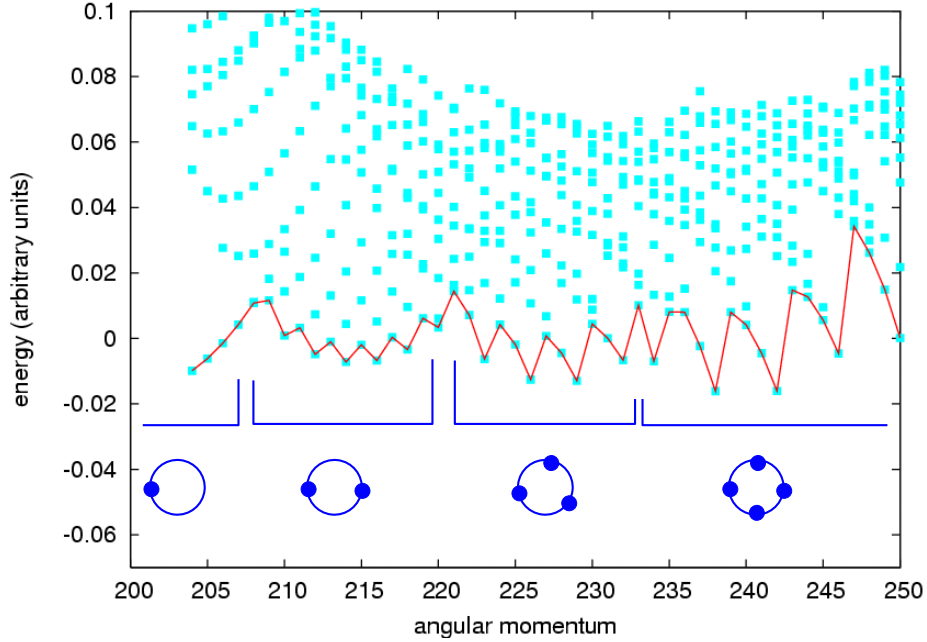
FIG. 1: Energy spectrum of 20 electrons in a quantum dot as a function of the angular momentum. The solid line connects the lowest energy states for each angular momentum and shows the characteristic oscillation caused by the vortex localization. The vortex textures are shown schematically below the regions for 1 to 4 vortices. (A smooth function of the angular momentum, $f(L) = A + BL + CL^2$, was subtracted from the energies to mimic the effect of the magnetic field on the total energy).
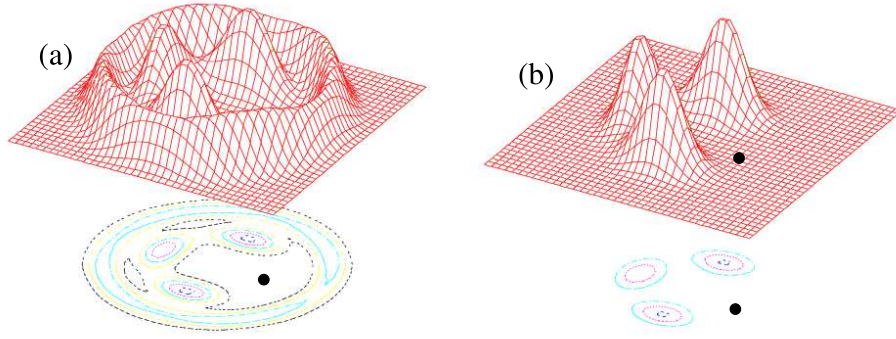
FIG. 2:   Vortex-vortex pair correlation function for the four-vortex state in a quantum dot with 20 electrons. The reference point is marked with a black dot. (a) shows the result for the calculation of 20 electrons with angular momentum 246 and (b) shows the result for the calculation of the conjugate system with 4 holes.
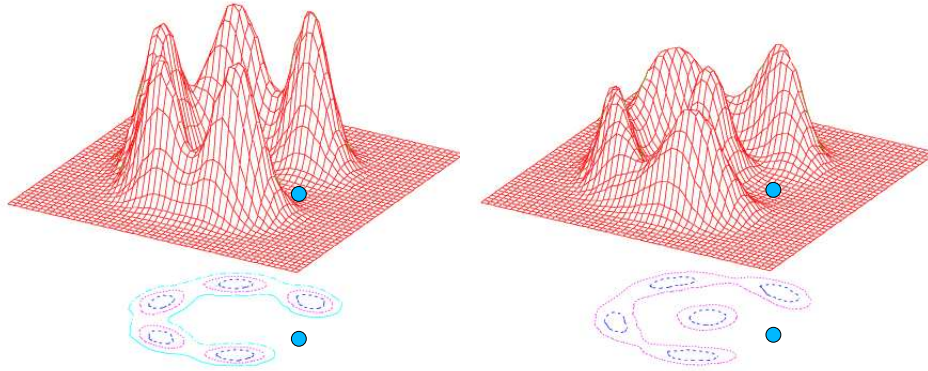

FIG. 3:   Vortex-vortex correlations of the two lowest energy states for 30 electron quantum dot at angular momentum 555. The ground state (a) shows 6 vortices localised in a ring, while the first excited state (b) shows a five vortex ring with one vortex in the centre.